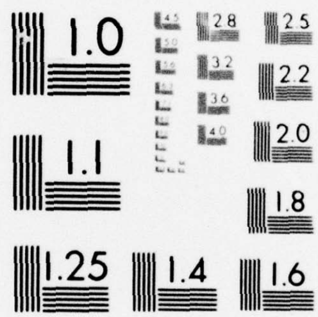
AD-A049 516 CALIFORNIA UNIV LOS ANGELES DEPT OF MATHEMATICS F/G 12/1
NORM PROPERTIES OF C-NUMERICAL RADII.(U)
1977 M GOLDBERG, E G STRAUS AFOSR-76-3046
UNCLASSIFIED AFOSR-TR-78-0028 NL

| OF |

AD
A049516

END
DATE
FILMED
2-78
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

NORM PROPERTIES OF C-NUMERICAL RADII

by

Moshe Goldberg [redacted] E. G. Straus*

Department of Mathematics
University of California
Los Angeles, California 90024

ABSTRACT. Given $n \times n$ complex matrices A, B, the C-numerical radius of A is the nonnegative quantity

$$r_C(A) \equiv \max\{|\mathrm{tr}(CU^*AU)| : U \text{ unitary}\}.$$

For $C = \mathrm{diag}(1,0,\ldots,0)$ it reduces to the classical numerical radius $r(A) = \max\{|x^*Ax| : x^*x = 1\}$. We show that r_C is a generalized matrix norm if and only if C is nonscalar and $\mathrm{tr}\, C \neq 0$. Next, we consider an arbitrary generalized matrix norm and characterize all constants $\nu > 0$ for which νN is multiplicative. A technique to obtain such ν is then applied to C-numerical radii with Hermitian C. In particular we find that νr is a matrix norm if and only if $\nu \geq 4$.

AMS (MOS) subject classification (1970). 15A60, 65F35

Key words and phrases. Numerical radius, semi-norms, generalized matrix norms, matrix norm, multiplicativity factors.

*The research of the first author was sponsored in part by the Air Force Office of Scientific Research, Air Force System Command, USAF, under Grant No. AFOSR-76-3046. The work of the second author was supported in part [redacted] NSF [redacted]-MPS 71-2884.

1. Introduction

Let $\mathbb{C}_{n\times n}$ be the algebra of $n \times n$ complex matrices and let $\mathcal{U}_n$ be its unitary group. Given $A, C \in \mathbb{C}_{n\times n}$, the C-numerical range of A is the compact set

$$W_C(A) = \{\mathrm{tr}(CU^*AU) : U \in \mathcal{U}_n\}.$$

This definition together with some properties of $W_C(A)$ were presented by the authors in [2].

It is not hard to see (compare [2], Lemma 9), that $W_C(A)$ is invariant under unitary similarities of A or C. Hence, if C is normal with eigenvalues γ_j, we easily find that

$$(1.1)\quad W_C(A) = W_{\mathrm{diag}(\gamma_1,\ldots,\gamma_n)}(A) = \left\{\sum_{j=1}^{n} \gamma_j x_j^* A x_j : \{x_j\} \in \Lambda_n\right\},$$

Λ_n being the set of orthonormal bases for $\mathbb{C}_n$. In particular, for $C = \mathrm{diag}(1,0,\ldots,0)$, we obtain the classical range

$$W(A) = \{x^*Ax : x^*x = 1\}.$$

Associated with the classical range is the numerical radius

$$r(A) = \max\{|z| : z \in W(A)\}.$$

Similarly, we define the C-numerical radius to be

$$r_C(A) = \max\{|z| : z \in W_C(A)\}.$$

The main purpose of this work is to study the norm properties of r_C. The situation is trivial for $n = 1$, so without further reference we assume troughout the paper that $n \geq 2$.

We use the following standard definitions.

(i) A mapping $A \to N(A)$ is a semi-norm on $\mathbb{C}_{n\times n}$, if for any $A, C \in \mathbb{C}_{n\times n}$ and $\alpha \in \mathbb{C}$,

$$N(A) \geq 0;$$

$$N(\alpha A) = |\alpha| N(A);$$

$$N(A + B) \leq N(A) + N(B).$$

(ii) A semi-norm is a <u>generalized matrix norm</u> if it is positive definite, that is,

$$N(A) > 0 \quad \text{for} \quad A \neq 0.$$

(iii) A generalized matrix norm is a <u>matrix norm</u> if it is (sub-) multiplicative, i.e., for all A, B,

$$N(AB) \leq N(A)N(B).$$

Without difficulty we obtain

THEOREM 1. <u>For any</u> C, r_C <u>is a semi-norm</u>.

The questions of definiteness and multiplicativity are much more complicated.

In Section 2 we characterize those C for which r_C is positive definite. We show that r_C is a generalized matrix norm if and only if C is not scalar and $\operatorname{tr} C \neq 0$. This result agrees with the well known fact that the classical radius r is a generalized matrix norm.

The classical radius is not multiplicative, [4]. Hence, in general, a C-radius cannot be expected to be a matrix norm.

In Section 3 we consider arbitrary generalized matrix norms, and characterize all positive constants ν for which νN is multiplicative. A technique of finding such <u>multiplicativity factors</u> is given by a theorem of Gastinel [1].

The above technique (aided by some combinatorial inequalities obtained in Section 4) is applied in Section 5 to find multiplicativity factors for C-numerical radii with Hermitian C. In particular we find that νr is a matrix norm if and only if $\nu \geq 4$.

Thanks are due to Alston Householder and to Robert Steinberg for helpful discussions.

2. Norm Characterization of C-radii.

THEOREM 2. r_C is a generalized matrix norm if and only if

(2.1) $\qquad C$ is nonscalar and $\operatorname{tr} C \neq 0$.

In the proof we use the following three lemmas in which A, C are given $n \times n$ matrices.

LEMMA 1. Let m be an integer with $1 \leq m < n$. If C leaves invariant all m-dimensional subspaces of $\mathbb{C}^n$, then C is scalar.

Proof. Since $m < n$, then each one-dimensional subspace of $\mathbb{C}^n$ is an intersection of subspaces of dimension m, which by hypothesis, are fixed by C. This implies that C fixes all one-dimensional subspaces of $\mathbb{C}_n$.

Now let $\{e_j\}_{j=1}^n$ be the standard basis of $\mathbb{C}^n$. By the preceding argument, there exist scalars $\lambda_1, \ldots, \lambda_n, \mu$, such that

$$Ce_j = \lambda_j e_j, \quad 1 \leq j \leq n,$$

and

$$C \sum_{j=1}^{n} e_j = \mu \sum_{j=1}^{n} e_j.$$

Hence, $\mu \Sigma e_j = \Sigma \lambda_j e_j$, and we conclude that $\lambda_j = \mu$, $1 \leq j \leq n$. Therefore,

$$Ce_j = \mu e_j, \quad 1 \leq j \leq n;$$

i.e., $C = \mu I$, and the lemma follows.

LEMMA 2. <u>If</u>

$$CU^*AU = U^*AUC \qquad \forall U \in \mathcal{U}_n,$$

<u>then either</u> A <u>or</u> C <u>are scalar</u>.

<u>Proof</u>. Suppose A is not scalar and let us prove that C is. Let λ be an eigenvalue of A with corresponding eigenspace $\mathcal{V}_\lambda$ of dimension m. Since A is not scalar, then

$$1 \leq m = \dim(\mathcal{V}_\lambda) < \dim(\mathcal{C}^n) = n.$$

Now, for arbitrary $U \in \mathcal{U}_n$, U^*AU also has λ as eigenvalue with corresponding eigenspace $U^*\mathcal{V}_\lambda$. Thus, for every vector $v \in U^*\mathcal{V}_\lambda$,

$$U^*AU(Cv) = C(U^*AUv) = C(\lambda v) = \lambda(Cv).$$

It follows that

$$Cv \in U^*\mathcal{V}_\lambda \qquad \forall v \in U^*\mathcal{V}_\lambda,$$

that is, C leaves $U^*\mathcal{V}_\lambda$ invariant. Since $\dim(\mathcal{V}_\lambda) = m$ and U^* is arbitrary, we find that C leaves invariant all m-dimensional subspaces of $\mathcal{C}^n$. Hence, by Lemma 1, C is scalar and the proof is complete.

LEMMA 3. <u>If</u>

$$\operatorname{tr}(CU^*AU) = \underline{\text{constant}} \qquad \forall U \in \mathcal{U}_n,$$

<u>then</u>

$$CU^*AU = U^*AUC \qquad \forall U \in \mathcal{U}_n.$$

<u>Proof</u>. Let S be skew-Hermitian; then $e^{\theta S}$ is unitary for all real θ, and so is $Ue^{\theta S}$. By hypothesis therefore,

$$f(\theta) \equiv \mathrm{tr}[C(Ue^{\theta S})^*A(Ue^{\theta S})] = \text{constant}, \qquad \theta \in \mathbb{R};$$

and consequently,

$$\frac{d}{d\theta}f(\theta) = \frac{d}{d\theta}\mathrm{tr}(Ce^{-\theta S}U^*AUe^{\theta S}) =$$

$$= \mathrm{tr}(Ce^{-\theta S}U^*ASe^{\theta S} - CSe^{-\theta S}U^*AUe^{\theta S}) = 0.$$

Evaluating the derivative at $\theta = 0$ we obtain

$$\mathrm{tr}(CU^*AUS - CSU^*AU) = 0;$$

hence for all skew-Hermitian S (and all unitary U),

$$\mathrm{tr}[(CU^*AU - U^*AUC)S] = 0.$$

Since every matrix B is a linear combination of skew-Hermitians*, the last identity implies

$$\mathrm{tr}[(CU^*AU - U^*AUC)B] = 0 \qquad \forall B \in \mathbb{C}_{n\times n}.$$

Thus,

$$CU^*AU - U^*AUC = 0,$$

and the lemma is proven.

<u>Proof of Theorem 2</u>. By Theorem 1, it suffices to show that (2.1) holds if and only if r_C is positive definite.

If C is scalar, namely $C = \lambda I$, then any $A \neq 0$ with $\mathrm{tr}\, A = 0$ gives

$$r_C(A) = |\lambda \,\mathrm{tr}\, A| = 0.$$

*For example, $B = S_1 - iS_2$ with $S_1 = \frac{1}{2}(B - B^*)$, $S_2 = \frac{i}{2}(B + B^*)$.

Also, if $\operatorname{tr} C = 0$, then

$$r_C(I) = |\operatorname{tr} C| = 0.$$

Thus, violation of (2.1) implies the indefiniteness of $r_C(\cdot)$.

Conversely, let (2.1) hold. If $r_C(A) = 0$, then by definition

$$\operatorname{tr}(CU^*AU^*) = 0 \qquad \forall U \in \mathcal{U}_n;$$

so by Lemma 3,

$$CU^*AU = U^*AUC \qquad \forall U \in \mathcal{U}_n.$$

By Lemma 2, therefore, either C or A are scalar, and since C is not, A is. Setting $A = \mu I$ we have

$$r_C(A) = |\mu \operatorname{tr} C| = 0,$$

and since $\operatorname{tr} C \neq 0$, then μ must vanish and the proof is established.

EXAMPLE 1. The k-numerical range, $1 \leq k \leq n$, was defined by Halmos [3, § 167] to be

$$W_k(A) = \{\operatorname{tr}(PA) : P \text{ orthonormal projection of rank } k\}.$$

We easily verify that

$$W_k(A) = W_{C_k}(A) \quad \text{where} \quad C_k = I_k \oplus 0_{n-k}.$$

Thus, the k-<u>numerical radius</u>

$$r_k(A) = \max\{|z| : z \in W_k(A)\},$$

is a generalized matrix norm if and only if $1 \leq k \leq n - 1$. In particular $r(A) = r_1(A)$ is a generalized norm while $r_n(A) = |\operatorname{tr} A|$ is not.

3. Multiplicativity Factors and Gastinel's Theorem

Given a semi-norm N on $\mathbb{C}_{n\times n}$ and a constant $\nu > 0$, then obviously

$$N_\nu \equiv \nu N$$

is a semi-norm too. Similarly, N is definite if and only if N_ν is. In any case the new norm may or may not be multiplicative. If it is, we say that ν is a <u>multiplicativity factor</u> of N.

A characterization of multiplicativity factors for generalized matrix norms is given in Theorem 4. We first prove, however, that indefinite nontrivial semi-norms have no multiplicativity factors.

THEOREM 3. <u>An indefinite semi-norm</u> N <u>on</u> $\mathbb{C}_{n\times n}$ <u>is multiplicative if and only if</u> $N \equiv 0$.

<u>Proof</u>. The trivial semi-norm is certainly multiplicative. So let N be indefinite and multiplicative, and let us show that $N \equiv 0$.

Since N is indefinite, then $N(A) = 0$ for some $A \neq 0$. Let $\alpha_{\ell k}$ be a nonvanishing entry of A, and denote by E_{ij} the matrix whose (i,j) element is 1 and the others are zero. Since

$$E_{i\ell}AE_{kj} = \alpha_{\ell k}E_{ij},$$

then by multiplicativity,

$$|\alpha_{\ell k}|N(E_{ij}) = N(\alpha_{\ell k}E_{ij}) \leq N(E_{i\ell})N(A)N(E_{kj}) = 0.$$

We conclude that

$$N(E_{ij}) = 0 \quad \forall 1 \leq i,\ j \leq n;$$

thus for any $B = (\beta_{ij}) \in \mathbb{C}_{n\times n}$,

$$N(B) = N\left(\sum_{i,j} \beta_{ij}E_{ij}\right) \leq \sum_{i,j} |\beta_{ij}|N(E_{ij}) = 0,$$

and the theorem follows.

THEOREM 4. <u>If</u> N <u>is a generalized matrix norm, then</u> ν <u>is a multiplicativity factor of</u> N (<u>i.e.</u>, N_ν <u>is a matrix norm) if and only if</u>

$$\nu \geq \nu_N \equiv \max_{A,B\neq 0} \frac{N(AB)}{N(A)N(B)}.$$

<u>Proof</u>. We write ν_N in the form

$$\nu_N = \max\{N(AB) : N(A) = N(B) = 1\},$$

and use a compactness argument to conclude that ν_N is well defined. It is clear then that $\nu_N > 0$.

Now, if $\nu \geq \nu_N$, then

$$N_\nu(AB) = \nu N(AB) \leq \nu\nu_N N(A)N(B) \leq \nu^2 N(A)N(B) = N_\nu(A)N_\nu(B);$$

hence N is multiplicative.

Conversely, if ν satisfies $0 < \nu < \nu_N$, we can find matrices A, B such that $\nu N(A)N(B) < N(AB)$. Thus we have

$$N_\nu(AB) = \nu N(AB) > \nu^2 N(A)N(B) = N_\nu(A)N_\nu(B),$$

and the proof is complete.

As an immediate consequence we have established

COROLLARY 1. <u>A generalized matrix norm</u> N_ν <u>is a matrix norm if and only if</u> $\nu_N \leq 1$.

In practice, Theorem 4 offers limited help since in general, ν_N is not easily evaluated. In the case of C-numerical radii, we were unable to find the optimal factor except for the classical radius.

An alternative way of finding multiplicativity factors is suggested by the following, somewhat stronger version of a theorem by Gastinel, [1].

THEOREM 5. <u>Let</u> N <u>be a semi-norm,</u> M <u>a matrix norm, and</u> $\eta \geq \xi > 0$ <u>constants such that</u>

$$\xi M(A) \leq N(A) \leq \eta M(A) \qquad \forall A \in \mathbb{C}_{n\times n}. \tag{3.1}$$

<u>Then,</u>

(i) N <u>is a generalized matrix norm.</u>

(ii) <u>For any</u> $\nu \geq \eta/\xi^2$, N_ν <u>is a matrix norm.</u>

(iii) <u>If</u> $\eta/\xi^2 \leq 1$, <u>then</u> N <u>is a matrix norm.</u>

<u>Proof</u>. Part (i) is trivial, and for part (ii) we should merely note that

$$N_\nu(AB) = \nu N(AB) \leq \nu\eta M(AB) \leq \nu\eta M(A)M(B)$$

$$\leq \frac{\nu\eta}{\xi^2} N(A)N(B) \leq \nu^2 N(A)N(B) = N_\nu(A)N_\nu(B).$$

Part (iii) then follows.

We recall, of course, that any two norms on $\mathbb{C}_{n\times n}$ are equivalent. Thus if N of Theorem 5 is known to be a matrix norm, then (3.1) always holds for suitable constants $\eta \geq \xi > 0$.

In Section 5 we use Theorem 5 to obtain multiplicativity factors for C-numerical radii with Hermitian C.

4. Some Combinatorial Inequalities

Let α_j, γ_j, $1 \leq j \leq n$, be scalars and consider the set

$$\mathcal{S}_\gamma(\alpha) = \left\{ \sum_{j=1}^{n} \gamma_j \alpha_{\sigma(j)} : \sigma \in S_n \right\},$$

S_n being the symmetric group. In this section we study bounds for the radius of $\mathcal{S}_\gamma(\alpha)$,

$$R_\gamma(\alpha) = \max\{|z| : z \in \mathcal{S}_\gamma(\alpha)\}.$$

A general remark is that all the involved quantities are invariant under rearrangements of the α_i and the γ_j, and under rotations of the form

$$(\alpha_1,\ldots,\alpha_n) \to e^{i\varphi}(\alpha_1,\ldots,\alpha_n),\ (\gamma_1,\ldots,\gamma_n) \to e^{i\psi}(\gamma_1,\ldots,\gamma_n)$$

which include, of course, change of sign. This fact will be repeatedly used in the proof of the following results.

LEMMA 4. *For any* $\alpha_j, \gamma_j \in \mathbb{C}$,

$$R_\gamma(\alpha) \geq \frac{1}{n}\left|\sum_{j=1}^{n} \alpha_j\right|\left|\sum_{j=1}^{n} \gamma_i\right|.$$

Proof. Let τ^i, $i = 1,2,\ldots,n$, be the powers of a nontrivial cyclic permutation in S_n. Since

$$\sum_{j=1}^{n} \gamma_j \alpha_{\tau^i(j)} \in \mathcal{S}_\gamma(\alpha), \qquad 1 \leq i \leq n,$$

then

$$R_\gamma(\alpha) \geq \left|\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\gamma_j\alpha_{\tau^i(j)}\right)\right|$$

$$= \left|\frac{1}{n}\sum_j \gamma_j \sum_i \alpha_{\tau^i(j)}\right| = \left|\frac{1}{n}\sum_j \gamma_j \sum_i \alpha_i\right|,$$

and the lemma holds.

LEMMA 5. <u>If</u> $\alpha_i \in \mathbb{R}$, $\gamma_j \in \mathbb{C}$, $1 \leq j \leq n$, <u>then</u>

$$R_\gamma(\alpha) \geq \frac{1}{2} \max_{i,j} |\alpha_i - \alpha_j| \max_{i,j} |\gamma_i - \gamma_j|.$$

<u>Proof</u>. Setting

$$\gamma_j = \lambda_j + i\mu_j, \qquad \lambda_j, \mu_j \in \mathbb{R},$$

we have

$$R_\gamma(\alpha) = \max_{\sigma \in S_n} \left| \sum_j \lambda_j \alpha_{\sigma(j)} + i \sum_j \mu_j \alpha_{\sigma(j)} \right|$$

$$\geq \max_{\sigma \in S_n} \left| \sum_j \lambda_j \alpha_{\sigma(j)} \right| = R_\lambda(\alpha)$$

Now, if the γ_j are equal, then the result is trivial; so by rotating and rearanging the γ_j, we may assume that

$$\max |\gamma_i - \gamma_j| = \gamma_1 - \gamma_n > 0.$$

It follows that

$$\lambda_1 - \lambda_n = \gamma_1 - \gamma_n = \max_{i,j} |\gamma_i - \gamma_j| \geq \max_{i,j} |\lambda_i - \lambda_j|.$$

Thus

$$\lambda_1 \geq \lambda_j \geq \lambda_n, \qquad 2 \leq j \leq n - 1,$$

so we may assume that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

We may also assume that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n.$$

Hence, observing that

$$s_1 = \sum \lambda_j \alpha_j, \qquad s_2 = \sum \lambda_j \alpha_{n-j}$$

are two points in $\mathfrak{g}_\gamma(\alpha)$, we have

$$R_\lambda(\alpha) \geq \frac{1}{2}\left|s_1 - s_2\right| = \frac{1}{2}\left|\lambda_1(\alpha_1 - \alpha_n) + \lambda_2(\alpha_2 - \alpha_{n-1}) + \cdots + \lambda_n(\alpha_n - \alpha_1)\right|$$

$$= \frac{1}{2}\left|(\lambda_1 - \lambda_n)(\alpha_1 - \alpha_n) + (\lambda_2 - \lambda_{n-1})(\alpha_2 - \alpha_{n-1}) + \cdots + (\lambda_{\left[\frac{n}{2}\right]} - \lambda_{\left[\frac{n}{2}\right]+1})(\alpha_{\left[\frac{n}{2}\right]} - \alpha_{\left[\frac{n}{2}\right]+1})\right|$$

$$\geq \frac{1}{2}(\lambda_1 - \lambda_n)(\alpha_1 - \alpha_n) = \frac{1}{2}\max|\gamma_i - \gamma_j|\max|\alpha_i - \alpha_j|,$$

and the lemma follows.

We are interested now in obtaining constants K_γ, which may depend on the γ_j but not on the α_j, such that

$$M_\gamma(\alpha) \geq K_\gamma \max|\alpha_j| \qquad \forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}. \tag{4.1}$$

THEOREM 6. <u>For given</u> $\gamma_j \in \mathbb{C}$, $1 \leq j \leq n$, <u>there exists a constant</u> $K_\gamma > 0$ <u>which satisfies</u> (4.1) <u>if and only if</u>

$$\gamma_j \text{ are not all equal and } \sum_j \gamma_j \neq 0. \tag{4.2}$$

<u>If</u> (4.2) <u>holds, then</u> (4.1) <u>is satisfied by the positive constant</u>

$$K_\gamma = \frac{|\sum_j \gamma_j| \cdot \max_{i,j}|\gamma_i - \gamma_j|}{2|\sum_j \gamma_j| + \max_{i,j}|\gamma_i - \gamma_j|}. \tag{4.3}$$

<u>Proof</u>. Suppose (4.2) is violated. If the γ_j are equal, we choose α_j not all equal, with $\sum \alpha_j = 0$; if $\sum \gamma_j = 0$, we take $\alpha_j = 1$, $1 \leq j \leq n$. In both cases $R_\gamma(\alpha) = 0$ but $\max|\alpha_j| > 0$; hence no positive K_γ satisfies (4.1).

Conversely, let (4.2) hold, and let K_γ be the constant specified in (4.3). We may assume that

$$\alpha_1 \geq \cdots \geq \alpha_n,$$

where in fact, by change of sign if necessary, it suffices to consider the cases

$$\alpha_1 \geq \cdots \geq \alpha_n \geq 0, \tag{4.4a}$$

and

$$\alpha_1 \geq \cdots \geq \alpha_k \geq 0 > \alpha_{k+1} \geq \cdots \geq \alpha_n \quad \text{with} \quad \max|\alpha_j| = \alpha_1. \tag{4.4b}$$

In case (4.4a) we write $\alpha_n = \theta\alpha_1$, $0 \leq \theta \leq 1$, and use Lemmas 4 and 5 to obtain, respectively,

$$R_\gamma(\alpha) \geq \frac{1}{n}|\Sigma\alpha_j|\,|\Sigma\gamma_j| > \alpha_n|\Sigma\gamma_j| = \theta\alpha_1|\Sigma\alpha_j| = \theta|\Sigma\gamma_j|\max|\alpha_j|$$

and

$$R_\gamma(\alpha) \geq \frac{1}{2}\max|\alpha_i - \alpha_j|\max|\gamma_i - \gamma_j| = \frac{1}{2}(\alpha_1 - \alpha_n)\max|\gamma_i - \gamma_j|$$
$$\geq \frac{1}{2}(1 - \theta)\max|\gamma_i - \gamma_j|\max|\alpha_j|.$$

We thus find that

$$R_\gamma(\alpha) \geq \max\left\{\theta|\Sigma\gamma_j|,\ \frac{1}{2}(1 - \theta)\max|\gamma_i - \gamma_j|\right\} \cdot \max|\alpha_j|.$$

The expressions in the above braces are functions of θ which describe straight lines with opposite slopes and intersection value K_γ. Hence, for any θ

$$\max\left\{\theta|\Sigma\gamma_j|,\ \frac{1}{2}(1 - \theta)\max|\gamma_i - \gamma_j|\right\} \geq K_\gamma,$$

and (4.1) follows.

In case (4.4b) we use Lemma 5 to find that

$$R_\gamma(\alpha) \geq \frac{1}{2}(\alpha_1 - \alpha_n)\max|\gamma_1 - \gamma_n| > \frac{1}{2}\max|\gamma_i - \gamma_j|\max|\alpha_j|.$$

Since

$$\frac{1}{2}\max|\gamma_i - \gamma_j| > K_\gamma,$$

then (4.1) holds again, and the theorem is proven.

The above result can be improved for certain classes of γ_j.

THEOREM 7. _If_ γ_j, $1 \leq j \leq n$, _are complex scalars of the same argument, then_ (4.1) _holds with_

$$K_\gamma = \frac{1}{2}\max_{i,j}|\gamma_i - \gamma_j|. \tag{4.5}$$

Proof. By change of argument and rearangement we may assume that

$$\gamma_1 \geq \cdots \geq \gamma_n \geq 0,$$

and that the α_j satisfy (4.4a) or (4.4b).

For (4.4a) we have

$$R_\gamma(\alpha) = \sum \gamma_j\alpha_j \geq \gamma_1\alpha_1 \geq \frac{1}{2}(\gamma_1 - \gamma_n)\alpha_1;$$

and for (4.4b), Lemma 5 yeilds

$$R_\gamma(\alpha) \geq \frac{1}{2}(\gamma_1 - \gamma_n)(\alpha_1 - \alpha_n) > \frac{1}{2}(\gamma_1 - \gamma_n)\alpha_1.$$

Thus,

$$R_\gamma(\alpha) \geq \frac{1}{2}\max|\gamma_i - \gamma_j|\max|\alpha_j|,$$

and the proof is complete.

Indeed, comparing K_γ of (4.5) with K_γ of (4.3), we realize that for the relevant γ_j, Theorem 6 provides a tighter lower bound for $R_\gamma(\alpha)$ than Theorem 5.

5. Multiplicative Hermitian Radii

As indicated previously, the purpose of this section is to obtain multiplicativity factors for C-numerical radii with Hermitian C.

LEMMA 6. Let A, C be normal matrices with eigenvalues α_j and γ_j, respectively. Then

$$r_C(A) = R_\gamma(\alpha).$$

Proof. Obviously, it suffices to show that

$$\text{conv } W_C(A) = \text{conv } \mathcal{S}_\gamma(\alpha).$$

Since $W_C(A)$ is invariant under unitary similarities of A and C, and since A and C are normal, then by (1.1),

$$W_C(A) = \left\{ \sum_{j=1}^{n} \gamma_j x_j^* \operatorname{diag}(\alpha_1, \ldots, \alpha_n) x_j : \{x_j\} \in \Lambda_n \right\}.$$

Thus, using the standard basis $\{e_j\}$, we find that every point in $\mathcal{S}_\gamma(\alpha)$ satisfies

$$\sum_j \gamma_j \alpha_{\sigma(j)} = \sum \gamma_j e_{\sigma(j)}^* \operatorname{diag}(\alpha_1, \ldots, \alpha_n) e_{\sigma(j)} \in W_C(A),$$

which gives us

$$\mathcal{S}_\gamma(\alpha) \subseteq W_C(A).$$

Conversely, take an arbitrary point,

$$\sum_j \gamma_j x_j^* \operatorname{diag}(\alpha_1, \ldots, \alpha_n) x_j \in W_C(A).$$

Since $x_j = (\chi_{j1},\ldots,\chi_{jn})^T$, $1 \leq j \leq n$, is an orthonormal basis, then $X \equiv [|\chi_{jk}|^2]$ is a doubly stochastic matrix. Doubly stochastic matrices are convex combinations of permutation matrices P_σ. Thus writing $X = \Sigma_\sigma \lambda_\sigma P_\sigma$ and

$$a \equiv (\alpha_1,\ldots,\alpha_n)^T, \qquad c \equiv (\gamma_1,\ldots,\gamma_n)^T,$$

we have

$$\sum_j \gamma_j x_j^* \operatorname{diag}(\alpha_1,\ldots,\alpha_n)x_j = \sum_{j,k} \gamma_j |\chi_{jk}|^2 \alpha_k = c^T X a =$$

$$= c^T \left[\sum_{\sigma \in S_n} \lambda \alpha_\sigma P_\sigma\right] a = \sum_\sigma \lambda_\sigma (c^T P_\sigma a) = \sum_\sigma \lambda_\sigma \left[\sum_j \gamma_j \alpha_{\sigma(j)}\right] \in \operatorname{conv} \mathcal{S}_\gamma(\alpha).$$

This yields

$$W_C(A) \subseteq \operatorname{conv} \mathcal{S}_\gamma(\alpha),$$

and the lemma follows.

LEMMA 7. Let C be normal with eigenvalues γ_j, let K_γ satisfy (4.1), and let

$$\|A\|_2 \equiv \max\{(x^*A^*Ax)^{1/2} : x^*x = 1\}$$

denote the spectral norm of A. Then

$$r_C(A) \geq K_\gamma \|H\|_2 \qquad \forall \text{Hermitian } H \in \mathbb{C}_{n\times n}.$$

Proof. For Hermitian H with eigenvalues α_j, we know that

$$\|H\|_2 = \max|\alpha_j|.$$

Since the α_j are real, we may use (4.1), and by Lemma 6

$$r_C(A) = R_\gamma(\alpha) \geq K_\gamma \max|\alpha_j| = K_\gamma \|A\|_2.$$

LEMMA 8. If C is Hermitian, then $r_C(A) = r_C(A^*)$.

Proof.

$$r_C(A) = \max_U |tr(CU^*AU)| = \max_U |tr(CU^*AU)^*| = \max_U |tr(U^*A^*UC)| = r_C(A^*).$$

LEMMA 9. If C is Hermitian with eigenvalues γ_j, and if K_γ satisfies (4.1), then

$$r_C(A) \geq \frac{1}{2} K_\gamma \|A\|_2 \qquad \forall A \in \underset{\sim}{C}_{n\times n}.$$

Proof. We write $A = \frac{1}{2}(H_1 - iH_2)$, where

$$H_1 = A + A^*, \quad H_2 = i(A - A^*),$$

are Hermitian. By Lemmas 7 and 8, and by Theorem 1,

$$\frac{1}{2} K_\gamma \|A\|_2 = \frac{1}{4} K_\gamma \|H_1 - iH_2\|_2 \leq \frac{1}{4} K_\gamma [\|H_1\|_2 + \|H_2\|_2] \leq \frac{1}{4}[r_C(H_1) + r_C(H_2)]$$

$$= \frac{1}{4}[r_C(A + A^*) + r_C(A - A^*)] \leq \frac{1}{2}[r_C(A) + r_C(A^*)] = r_C(A),$$

and the proof is complete.

LEMMA 10. If C is normal with eigenvalues γ_j, then

$$r_C(A) \leq \sum_j |\gamma_j| \|A\|_2 \qquad \forall A \in \underset{\sim}{C}_{n\times n}.$$

Proof. By (1.1) we have

$$r_C(A) = \max\left\{\left|\sum_j \gamma_j x_j^* A x_j\right| : \{x_j\} \in \Lambda_n\right\};$$

and since $|x^*Ax| \leq \|A\|_2$ for any unit vector x, the lemma follows.

THEOREM 8. Let C be Hermitian, nonscalar, with $tr\, C \neq 0$ and eigenvalues γ_j. Then, for any ν with

$$\nu \geq 4 \sum_j |\gamma_j| \left[\frac{2|\sum_j \gamma_j| + \max_{i,j}|\gamma_i - \gamma_j|}{|\sum_j \gamma_j| \cdot \max_{i,j}|\gamma_i - \gamma_j|}\right]^2,$$

the (Hermitian) numerical radius $\nu r_C \equiv r_{\nu C}$ is a matrix norm.

Proof. Since C is nonscalar, the γ_j are not all equal; and since $\operatorname{tr} C \neq 0$, then $\Sigma \gamma_j \neq 0$. Thus, by Theorem 6, the inequality in (4.1) is satisfied by the positive constant K_γ of (4.3). By Lemmas 9 and 10 we have therefore,

$$\frac{1}{2} \frac{|\Sigma \gamma_j| + \max|\gamma_i - \gamma_j|}{2|\Sigma \gamma_j| + \max|\gamma_i - \gamma_j|} \|A\|_2 \leq r_C(A) \leq \Sigma |\gamma_j| \, \|A\|_2 \qquad \forall A \in \mathbb{C}_{n\times n},$$

and Theorem 5 completes the proof.

For Hermitian definite C, we improve Theorem 6 as follows.

THEOREM 9. Let C be Hermitian nonnegative (nonpositive) definite. If C is nonscalar with eigenvalues γ_j, then for every ν with

$$\nu \geq \frac{16 \sum_i |\gamma_j|}{\max_{i,j} |\gamma_i - \gamma_j|},$$

$\nu r_C \equiv r_{\nu C}$ is a matrix norm.

Proof. Since C is Hermitian definite, the γ_j are of the same sign and by Theorem 7, K_γ of (4.5) satisfies (4.1). Lemmas 9 and 10 yield now,

$$(5.1) \qquad \frac{1}{4} \max|\gamma_i - \gamma_j| \, \|A\|_2 \leq r_C(A) \leq \Sigma |\gamma_j| \, \|A\|_2 \qquad \forall A \in \mathbb{C}_{n\times n}.$$

Since C is nonscalar, the γ_j are not all equal; so $\max|\gamma_i - \gamma_j| > 0$, and Theorem 5 completes the proof.

EXAMPLE 2. We recall the definition of the k-numerical radius r_k. By Theorem 7, we find that νr_k, $1 \leq k \leq n-1$, is a matrix norm if $\nu \geq 16k$.

Example 2 implies that $\nu \geq 16$ is a multiplicativity factor for the classical radius r. The optimal factor, ν_r, is given in the following result.

THEOREM 10. νr is a matrix norm if and only if $\nu \geq 4$; that is $\nu_r = 4$.

Proof. It is well known, [3, §162], that

$$\tfrac{1}{2}\|A\|_2 \leq r(A) \leq \|A\|_2 \qquad \forall A \in \mathbb{C}_{n\times n}.$$

Thus, by Theorem 5, $\nu \geq 4$ is a multiplicativity factor for r, and by Theorem 4, $\nu_r \leq 4$.

To show that $\nu_r \geq 4$, consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus I_{n-2}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}.$$

A simple calculation shows that

$$r(A) = r(B) = \tfrac{1}{2}, \qquad r(AB) = 1.$$

Hence

$$r_\nu(AB) \leq r_\nu(A) r_\nu(B)$$

if and only if $\nu \geq 4$, and the theorem follows.

REFERENCES

1. N. Gastinel, Linear Numerical Analysis, Academic Press, 1970.
2. M. Goldberg and E. G. Straus, Elementary inclusion relations for generalized numerical ranges, Linear Algebra Appl., Vol. 18 (1977) 1-24.
3. P. R. Halmos, A Hilbert Space Problem Book, Van Nostrand, 1967.
4. C. Pearcy, An elementary proof for the power inequality for numerical radius, Mich. Math. J., Vol. 13 (1966) 289-291.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 78-0028	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): NORM PROPERTIES OF C-NUMERICAL RADII	5. TYPE OF REPORT & PERIOD COVERED: Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Moshe Goldberg and E. G. Straus	8. CONTRACT OR GRANT NUMBER(s): AFOSR-76-3046
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of California, Department of Mathematics, Los Angeles, CA 90024	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A3
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: 1977
	13. NUMBER OF PAGES: 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Numerical radius; semi-norms; generalized matrix norms; matrix norm; multiplicativity factors.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Given $n \times n$ complex matrices A, B, the C-numerical radius of A is the nonnegative quantity

$$r_C(A) \equiv \max\{|\mathrm{tr}(CU^*AU)| : U \text{ unitary}\}.$$

For $C = \mathrm{diag}(1,0,\dots,0)$ it reduces to the classical numerical radius

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(con't) 20. Abstract
$r(A) = \max\{|x^*Ax| : x^*x = 1\}$. We show that r_C is a generalized matrix norm if and only if C is nonscalar and $\operatorname{tr} C \neq 0$. Next, we consider an arbitrary generalized matrix norm and characterize all constants $\nu > 0$ for which νN is multiplicative. A technique to obtain such ν is then applied to C-numerical radii with Hermitian C. In particular we find that νr is a matrix norm if and only if $\nu \geq 4$.